Models for Inexact Reasoning

Fuzzy Logic – Lesson 6
Inference from Conditional Fuzzy Propositions

Master in Computational Logic
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Inference in Classical Logic

• Inference rules in classical logic are based on tautologies
• Three classical inference rules:
  – Modus Ponendo Ponens (Modus Ponens)
    • Latin for “The way that affirms by affirming”
  – Modus Tollens (Modus Tollendo Tollens)
    • Latin for “The way that denies by denying”
  – Hypothetical Syllogism

\[
\begin{align*}
  p \rightarrow q & \quad p \rightarrow q & \quad p \rightarrow q \\
  p & \quad \neg q & \quad q \rightarrow r \\
  q & \quad \neg p & \quad p \rightarrow r
\end{align*}
\]
Generalization of Inference Rules

- Classical inference rules can be generalized in the context of fuzzy logic
- Generalized inference rules provide a framework to facilitate approximate reasoning
- Generalized versions of MP, MT and HS
- Generalization based on:
  - Fuzzy relations
  - The compositional rule of inference
The Compositional Rule of Inference

• Let $R$ be a crisp relation defined over $X \times Y$
• Given a value $x = u$ it is possible to infer that $y \in B = \{y \in Y | <u, y> \in R\}$
• Moreover, given a set $A \subseteq X$ we can infer that $y \in B = \{y \in Y | <x, y> \in R, x \in A\}$
The Compositional Rule of Inference

• Now assume that $R$ is a fuzzy relation on $X \times Y$
• Let $A'$ be a fuzzy set defined over the elements of the crisp set $X$
• It is possible to infer a fuzzy set $B'$ defined over the elements of the crisp set $Y$

\[
\mu_{B'}(y) = \sup_{x \in X} \left[ \min \left( \mu_A(x), R(x, y) \right) \right]
\]
The Compositional Rule of Inference

- When dealing with discrete sets the CRI can be also expressed in matrix form.
- Resorting to the definition of the composition of fuzzy relations we have:

\[(B') = (A') \circ (R)\]

- \(A'\) is the vector associated to fuzzy set \(A'\).
- \(R\) is the matrix associated to fuzzy relation \(R\).
- \(B'\) is the vector associated to the inferred fuzzy set \(B'\).
The Generalized Modus Ponens

• Let consider the following conditional fuzzy proposition:

\[ \text{p: "If } X \text{ is } A, \text{ then } Y \text{ is } B" \]

• Note that a fuzzy relation \( R \) is embedded in \( p \)
  – An implication relationship between fuzzy sets \( A, B \)

\[ R(x, y) = J(\mu_A(x), \mu_B(y)) \]

• The operator \( J(\cdot, \cdot) \) denotes a fuzzy implication
The Generalized Modus Ponens

• Now, we are given a second proposition
  \( q: “X \text{ is } A’” \)
• Viewing \( p \) as a rule and \( q \) as a fact we have:
• Applying the CRI on \( R’ \) and \( A’ \) we can conclude that:

  Rule: If \( X \) is \( A \), then \( Y \) is \( B \)
  Fact: \( X \) is \( A’ \)

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  Conclusion: \( Y \) is \( B’ \)
Example

- X = {x_1, x_2, x_3}, Y = {y_1, y_2}
- A = .5/x_1 + 1/x_2 + .6/x_3, B = 1/y_1 + .4/y_2
- A' = .6/x_1 + .9/x_2 + .7/x_3
- Use the compositional rule of inference to derive a conclusion in the form “Y is B’”
- Use Lukasiewicz’s implication
  - J(x, y) = min(1, 1-a+b)
The Generalized Modus Tollens

• The generalized modus tollens is expressed as:

Rule: If X is A, then Y is B
Fact: Y is B'

Conclusion: X is A'

• In this case, the CRI is expressed as follows:

\[ \mu_{A'}(x) = \sup_{y \in Y} \left[ \min(\mu_{B'}(y), R(x, y)) \right] \]

• Or in matrix form:

\[ A' = B' \circ R \]
Example

- \( X = \{x_1, x_2, x_3\}, \ Y = \{y_1, y_2\} \)
- \( A = .5/x_1 + 1/x_2 + .6/x_3, \ B = 1/y_1 + .4/y_2 \)
- \( B' = .9/y_1 + .7/y_2 \)
- Use the compositional rule of inference to derive a conclusion in the form “\( X \) is \( A' \)”
- Use Lukasiewicz’s implication
  \[ J(x, y) = \min(1, 1-a+b) \]
The Generalized Hypothetical Syllogism

• The HS can be expressed as follows:

  Rule 1: If X is A, then Y is B
  Rule 2: If Y is B, then Z is C

  Conclusion: If X is A, then Z is C

• In this case, we can say that the HS holds if:

  \[ R_3(x, z) = \sup_{y \in Y} \left[ \min \left( R_1(x, y), R_2(y, z) \right) \right] \]

• Or in matrix form:

  \[ R_3 = R_1 \circ R_2 \]
Example

- X = \{x_1, x_2, x_3\}, Y = \{y_1, y_2\}, Z = \{z_1, z_2\}
- A = 0.5/x_1 + 1/x_2 + 0.6/x_3, B = 1/y_1 + 0.4/y_2, C = 0.2/z_1 + 1/z_2

- Determine whether or not the HS holds in this case
- Use the following implication:

\[ j(a, b) = \begin{cases} 
1 & a \leq b \\
\frac{1}{b} & a > b 
\end{cases} \]
Exercise (Homework)

- Determine whether or not the HS holds for the case presented in the previous slide
- Use the Lukasiewicz’s implication