Models for Inexact Reasoning

Fuzzy Logic – Lesson 2
Fuzzy Propositions

Master in Computational Logic
Department of Artificial Intelligence
Fuzzy Propositions

• Main difference between classical propositions and fuzzy propositions:
  – The range of their truth values: [0, 1]

• We will focus on the following types of propositions:
  – Unconditional and unqualified propositions
    • “The temperature is high”
  – Unconditional and qualified propositions
    • “The temperature is high is very true”
  – Conditional and unqualified propositions
    • “If the temperature is high, then it is hot”
  – Conditional and qualified propositions
    • “If the temperature is high, then it is hot is true”
Unconditional and Unqualified Fuzzy Propositions

• The canonical form $p$ of fuzzy propositions of this type is:

$$p : V \text{ is } F$$

• $V$ is a variable that takes values $v$ from some universal set $E$

• $F$ is a fuzzy set on $E$ that represents a given imprecise predicate: tall, expensive, low, etc.

• Example:

$$p: \text{temperature (V) is high (F)}$$
Unconditional and Unqualified Fuzzy Propositions

• Given a particular value $V = v$, this individual belongs to $F$ with membership grade $\mu_F(v)$

• The membership grade is interpreted as the degree of truth $T(p)$ of proposition $p$

$$T(p) = \mu_F(v)$$

• Note that $T$ is also a fuzzy set on $[0, 1]$  
  – It assigns truth value $\mu_F(v)$ to each value $v$ of variable $V$
Example

- Let $V$ be the air temperature (in °F) at some place on the Earth
- Let $F$ be the fuzzy set that represents the predicate “high” (temperature)
Example

• The degree of truth $T(p)$ depends on:
  – The actual value of the temperature
  – The given definition (meaning) of predicate high

• Let us suppose that the actual temperature is 85 F
Unconditional and Unqualified Fuzzy Propositions

• Role of function $T$:
  – Provide us with a “bridge” between fuzzy sets and fuzzy propositions

• Note that $T$ is numerically trivial for unqualified propositions
  – The values are identical to those provided by the fuzzy membership function
  – This does not happen when dealing with qualified propositions (more complex $T(p)$ functions)
Unconditional and Qualified Propositions

- Two different canonical forms to represent these propositions:
  \[ p : \text{V is F is S} \]
  \[ p : \text{Pr(V is F) is P} \]

- V and F have the same meaning as in previous slides
- S is a fuzzy truth qualifier
- P is a fuzzy probability qualifier
Truth-qualified Propositions

• There are different truth qualifiers
  – Unqualified propositions are special truth-qualified propositions (S is assumed to be true)
Truth-qualified Propositions

• In general, the degree of truth $T(p)$ of any truth-qualified proposition $p$ is:

$$T(p) = \mu_S(\mu_F(v)), \quad \forall v \in E$$

• The membership function $\mu_G = \mu_s \circ \mu_F$ can be interpreted as the unqualified proposition “$V$ is $G$”
Example

• Proposition “Tina is young is very true”
  – Predicate: young
  – Qualifier: very true

• Let us suppose that the age of Tina is 26
Probability-qualified Proposition

- There are different probability qualifiers:
Probability-qualified Propositions

• Given a probability distribution $f$ on $V$, we can define the probability of a fuzzy proposition:

$$\Pr(V \text{ is } F) = \sum_{v \in V} f(v) \cdot \mu_F(v)$$

• We calculate the degree $T(p)$ to which a proposition of the form $[\Pr(V \text{ is } F) \text{ is } P]$ is true as:

$$T(p) = \mu_p(\Pr(V \text{ is } F))) = \mu_p\left(\sum_{v \in V} f(v) \cdot \mu_F(v)\right)$$
Example

• $p$: Pro(temperature is around 75 F) is likely
• The predicate “around 75 F” is represented by the following membership function:
Example

• The probability distribution obtained from relevant statistical data over many years is:

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<th>70</th>
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• Then, we calculate $\Pr(\text{temperature is around } 75 \text{ F})$ as follows:

$$\Pr(t \text{ is around } 75^\circ F) = .01 \times .25 + .04 \times .5 + .11 \times .75 + .15 \times 1 + .21 \times 1$$

$$+ .16 \times 1 + .14 \times .75 + .11 \times .5 + .04 \times .25 = .8$$
Example

• Now, we use the fuzzy qualifier to calculate the truth value of the predicate:
Conditional and Unqualified Propositions

• The canonical form $p$ of fuzzy propositions of this type is:

$$ p : \text{if } X \text{ is } A, \text{ then } Y \text{ is } B $$

• $X, Y$ are variables in universes $E_1$ and $E_2$
• $A, B$ are fuzzy sets on $X, Y$
• These propositions may also viewed as propositions of the form:

$$ \langle X, Y \rangle \text{ is } R $$
Conditional and Unqualified Propositions

• R is a fuzzy set on $X \times Y$ defined as:

$$R(x, y) = \mathcal{J} [\mu_A(x), \mu_B(y)]$$

• $\mathcal{J}$ represents a suitable fuzzy implication
  – There are many of them
• In our examples we will use the Lukasiewicz implication:

$$\mathcal{J}(a, b) = \min(1, 1 - a + b)$$
Example

• $A = \frac{1}{x_1} + \frac{.8}{x_2} + \frac{1}{x_3}$
• $B = \frac{.5}{y_1} + \frac{1}{y_2}$
• Łukasiewicz implication $\vdash J = \min(1, 1-a+b)$
• ¿R?

$T(p) = 1$ when $(X = x_1 \text{ and } Y = y_1)$
• $T(p) = .7$ when $(X = x_2 \text{ and } Y = y_1)$
• and so on...
Conditional and Qualified Propositions

• Propositions of this type can be characterized by two different canonical forms:

\[ p : (\text{if } X \text{ is } A, \text{ then } Y \text{ is } B) \text{ is } S \]
\[ p : \Pr(X \text{ is } A | Y \text{ is } B) \text{ is } P \]

• S is a fuzzy truth qualifier
• P is a fuzzy probability qualifier
• \( \Pr(X \text{ is } A | Y \text{ is } B) \) is a conditional probability
• Methods introduced before can be combined to deal with propositions of this type
Exercise

• Fuzzy Predicates
  - $\mu_{\text{tall}} = 0.5/160 + 0.75/170 + 1/180 + 1/190$
  - $\mu_{\text{short}} = 1/150 + 1/160 + 0.75/170 + 0.5/180$

• Statistics

<table>
<thead>
<tr>
<th>Couple</th>
<th>Husband (height, cm)</th>
<th>Wife (height, cm)</th>
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<tr>
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• Calculate the truth value associated to the following proposition:
  “Pr(husband is tall | wife is short) is likely”

• Use the Lukasiewicz implication
Homework

• Exercise:
  – Describe the method to deal with conditional and truth-qualified propositions
  – Provide a concrete example